On solutions of a discretized Heat equation in discrete Clifford analysis

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Abstract

The main purpose of this paper is to study solutions of the Heat equation in the setting of discrete Clifford Analysis. More precisely we consider this equation with discrete space and continuous time. Thereby we focus on representations of solutions by means of dual Taylor series expansions. Furthermore we develop a discrete convolution theory, apply it to the inhomogeneous Heat equation and construct solutions for the related Cauchy problem by means of Heat polynomials.

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1 Introduction

Clifford analysis is a natural higher-dimensional generalization of complex function theory and based on the study of properties of null solutions of the so-called Dirac operator which generalizes the $\partial$ operator. The null solutions of the Dirac operator are called monogenic functions. Because the Dirac operator factorizes the Laplacian Clifford analysis is also a refinement of classical harmonic analysis.

A very new branch of Clifford analysis is discrete Clifford analysis. The interest in a discrete theory is driven by computational physics and numerical computations. Because Clifford analysis is a generalization of complex function theory we look first at complex discrete functions.

Discrete monodiffric functions are defined on square lattices where introduced by J. Ferrand [10] and also [9]. They are based on the discretization of the Cauchy-Riemann equations

\[
f_{m,n+1} - f_{m+1,n} = i(f_{m+1,n+1} - f_{m,n}).
\]

A discrete function theory should mirror the properties of the continuous one, i.e. discrete holomorphic functions should be discrete harmonic and one needs a discrete

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version of the Cauchy integral. A major problem consists in the fact that discrete holomorphic functions do not form an algebra.

There are already a lot of contributions regarding the construction of discrete Dirac operators [11, 12, 14, 23, 24, 15, 25] but these constructions are based on potential-theoretic arguments [18, 13, 11, 20, 21]. Classical function theory is also built up by polynomials. All the theories mentioned above don’t give basic polynomials, a Fischer decomposition or a Taylor series.

An alternative approach is based on one type of difference operator [22], more precisely either a forward or backward difference operator, and leads to basic polynomials and a Fischer decompositions. But here it is not possible to factorize the star Laplacian, because such a factorization needs both types of difference.

In [2] the problem was considered from a more abstract point of view. Differentiating means lowering the power of a polynomial and multiplication with a variable raises the power. This gives a way to apply umbral calculus [19]. The connection between lowering operator and raising operator is given by the so-called “Weyl relations”

\[ \partial_j x_j - x_j \partial_j = 1 \text{ or, applied to a function } f, \quad \partial_j (x_j f(x)) - x_j \partial_j f(x) = f(x). \]

Then the corresponding theory is based on a duality argument, the so-called Fischer duality.

Even though this looks like a possible approach there is a major obstacle. While forward and backward differences are mutually commuting with each other, this is no longer true for the corresponding vector variables. The idea is to split the basis elements \( e_k \) into two new basis elements \( e^+_k \) and \( e^-_k \) and construct an appropriate discrete Dirac operator involving both forward and backward difference operators. Because the raising operators do not commute with each other, the standard Weyl relations had been replaced by a type of “skew” Weyl relations.

The success of this theory is proven by a couple of papers where it was possible to construct an Euler operator, discrete homogeneous polynomials, a discrete Cauchy-Kovalevskaya extension theorem [3], Fueter polynomials [6], discrete Taylor series [4], discrete Fourier transform and distributions [5] in a discrete Clifford setting.

In this paper we study the Cauchy problem for the Heat equation

\[ \frac{\partial}{\partial t} u(x, t) - \Delta_x u(x, t) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n \]

as a first step towards applications of the discrete function theory. A standard method to solve the continuous problem is to use a fundamental solution [8]. The function

\[ \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \]

with \( x \in \mathbb{R}^n \) and \( t > 0 \) is the fundamental solution of the Heat equation, which is also denoted as Heat kernel or Gaussian distribution. For all fixed \( t > 0 \) it has the properties of a probability distribution. For an initial function \( u_0 \) that belongs to the Schwartz space \( S = S(\mathbb{R}^n) \) of rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^n \), the solution \( u(x, t) \) is found by convolution of \( u_0 \) with the fundamental solution.
Drawing inspiration from the continuous setting, we study the discrete Gaussian and the discrete Gaussian distribution as fundamental solutions of the discrete heat equation. A function of the form $e^{-a|x|^2}$, $a > 0$, is usually called a “Gaussian.” It is easily seen that such a function is bell-shaped and has a maximum at 0. Another important property is the function $f(x) = e^{-\frac{|x|^2}{4}}$ is equal to their Fourier transform, i.e. $\hat{f}(\xi) = e^{-\frac{|\xi|^2}{2}}$. Therefore such functions have a rich history within mathematics and appear in a wide variety of settings.

It is obvious that a discrete analog of the Gaussian should have similar properties. A simple answer is to sample the continuous Gaussian. Unfortunately, this is not a good discrete analog. A better choice is the discrete Gaussian kernel:

$$G_t = e^{-2t} \text{BesselI}(|x|, 2t),$$

where BesselI denotes the modified Bessel functions of the first kind of integer order. With the aid of the discrete Gaussian kernel we construct a discrete fundamental solution allowing us to solve the Cauchy problem.

The paper is organized as follows: we start with an introduction. To keep the paper self-explanatory, Section 2 contains the preliminaries about the discrete framework. In Section 3 we introduce the discrete Gaussian distribution and compare its density function with the continuous counterpart. It turns out that the density function shows the well-known bell-shape with a maximum at the origin. In Section 4 we calculate the fundamental solution for the discrete Heat equation, based on a dual Taylor series expansion, showing an intrinsic connection with the discrete Gaussian. In Section 5 we give the basic properties of discrete convolution theory. In Section 6 we apply this for the solutions of the discrete Heat equation and we consider some special solutions which could be considered as basic building blocks. In Section 7, we construct the solutions of the Heat equation by means of a series expansion in the Heat polynomials.

In this paper, we considered a discrete version of the Heat equation at which space is discrete and time continuous. The fundamental solution is given by the discrete Gauss distribution, allowing us to construct solutions of the inhomogeneous Heat equation. We furthermore constructed solutions of the Cauchy problem for the Heat equation by means of a series expansion in the Heat polynomials.

## 2 Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidian space with orthonormal basis $e_j$, $j = 1, \ldots, n$ and consider the Clifford algebra $\mathbb{R}_{0,n}$ over $\mathbb{R}^n$. Passing to the so called discrete Hermitian setting, we imbed the Clifford algebra $\mathbb{R}_{0,n}$ into the bigger complex one $\mathbb{C}_{2n}$, i.e. the underlying vector space is of twice the dimension, and introduce forward and backward basis elements $e_j^\pm$ satisfying the following anti-commutator rules, cf. [1]:

$$e_j^- e_\ell^- + e_\ell^- e_j^- = e_j^+ e_\ell^+ + e_\ell^+ e_j^+ = 0 \quad \text{and} \quad e_j^+ e_\ell^- + e_\ell^- e_j^+ = \delta_{j,\ell}.$$

The connection to the original basis $e_j$ is given by $e_j^+ + e_j^- = e_j$. Observe that this implies $e_j^2 = 1$ in contrast to the usual Clifford setting where traditionally $e_j^2 = -1$ is chosen.
This approach is motivated due to the aim of constructing a discrete Dirac operator factorizing the discrete Laplacian. Consider therefore the standard equidistant lattice $\mathbb{Z}^n$, i.e. the coordinates of a Clifford vector $\tilde{x}$ take only integer values. The partial derivatives $\partial_{x_j}$ used in Euclidian Clifford analysis are replaced by forward and backward differences $\Delta_j^\pm$, $j = 1, \ldots, n$, acting on Clifford valued functions $f$:

$$\Delta_j^+[f] = f(\cdot + e_j) - f(\cdot) \quad \text{and} \quad \Delta_j^-[f] = f(\cdot) - f(\cdot - e_j).$$

These operators are seen as lowering operators. With respect to the $\mathbb{Z}^n$-neighbourhood of $\tilde{x}$ the usual definition of the discrete Laplacian reads

$$\Delta^*[f](\tilde{x}) = \sum_{j=1}^n \Delta_j^+ \Delta_j^-[f] = \sum_{j=1}^n \left( f(\tilde{x} + e_j) - f(\tilde{x} - e_j) \right) - 2nf(\tilde{x}).$$

This operator is also known as “Star Laplacian”. An appropriate definition of a discrete Dirac operator $\partial$ factorizing $\Delta^*$ is obtained by means of the forward and backward basis elements, more precisely by

$$\partial = \sum_{j=1}^n (e_j^+ \Delta_j^+ - e_j^- \Delta_j^-).$$

In order to receive an analogue of the classical Weyl relations $\partial_{x_j} x_k - x_k \partial_{x_j} = \delta_{jk}$, we introduce raising operators $X_j^\pm$ being characterized by their interaction with the corresponding lowering operators $\Delta_j^\pm$ according to the skew Weyl relations, cf. [2]

$$\Delta_j^+ X_j^+ - X_j^- \Delta_j^- = \Delta_j^- X_j^+ - X_j^+ \Delta_j^+ = 1.$$

In the discrete vector variable $\xi = \sum_{j=1}^n (e_j^+ X_j^- + e_j^- X_j^+)$, the operator components $X_j^\pm$ occur intertwined with the forward and backward basis elements $e_j^\pm$. Observe that the co-ordinate variables $\xi_j = e_j^+ X_j^- + e_j^- X_j^+$ and $\partial_j = e_j^+ \Delta_j^+ + e_j^- \Delta_j^-$ of $\xi = \sum_{j=1}^n \xi_j$ and $\partial = \sum_{j=1}^n \partial_j$ thus satisfy the following anti-commutator relations

$$\partial_j \xi_j - \xi_j \partial_j = 1 \quad \text{and} \quad \partial_\ell \xi_j + \xi_j \partial_\ell = 0, \ \ell \neq j, \ j, \ell = 1, \ldots, n$$

which imply

$$\partial_j \xi_j^k[1] = k \xi_j^{k-1}[1], \ \partial_\ell \xi_j^k[1] = 0 \text{ and } \partial_j \xi_j^{k_1} \xi_\ell^{k_2}[1] = k_1 \xi_j^{k_1-1} \xi_\ell^{k_2}[1], \ j \neq \ell.$$  

The $\xi_j$ mutually anti-commute ($\xi_j \xi_k = -\xi_k \xi_j$ for $j \neq k$) as do the differences $\partial_j$.

The natural powers $\xi_j^k[1]$ of the operator $\xi_j$ acting on the ground state 1 are the basic discrete homogeneous polynomials of degree $k$ in the variable $x_j$, replacing the basic
powers $x_j^k$ in the continuous setting and constituting a basis for all discrete polynomials, cf. [3]. The odd and even powers are explicitly given by $\xi_j[1](x_j) = x_j (e_j^+ + e_j^-)$ and

$$
\xi_j^{2n}[1](x_j) = (x_j^2 + n x_j (e_j^+ e_j^- - e_j^- e_j^+)) \prod_{i=1}^{n-1} (x_j^2 - i^2)
$$

$$
\xi_j^{2n+1}[1](x_j) = x_j \prod_{i=1}^{n} (x_j^2 - i^2) (e_j^+ + e_j^-), \ n \geq 1.
$$

Observe that

$$
\xi_j^k[1](x_j) = 0 \text{ if } k \geq 2 |x_j| + 1. \tag{1}
$$

By means of these operators a discrete function with compact support $\Omega$ can be expanded in a Taylor series about the point zero

$$
f(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha_1! \cdots \alpha_n!} \xi_{\alpha_1} \cdots \xi_{\alpha_n}[1](x) \partial_{\alpha_n} \cdots \partial_{\alpha_1} f(0).
$$

Therefore we assume that $\Omega$ contains the origin and for all $x \in \Omega$ the inclusion $\{y_j : |y_j| \leq |x_j|\} \subset \Omega$ holds, since each difference of order $\alpha_j$ requires $\lceil \alpha_j/2 \rceil$ extra points in forward and backward direction where $f$ needs to be defined, cf. [4]. Due to (1) such a series for functions on bounded domains always converges. If $f$ is additionally left discrete monogenic in $\Omega$ and defined on the set $\bigcup_{x \in \Omega} \{y : |y| \leq |x| + h\}$, then its Taylor series can be expressed in terms of discrete Fueter polynomials $V_{\alpha_1,\ldots,\alpha_n}$ which form a basis for the space of discrete spherical monogenics with degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$$
f(x) = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} V_{\alpha_1,\ldots,\alpha_n} \partial_{\alpha_n} \cdots \partial_{\alpha_1} f(0) \right).
$$

For the converse assertion and further details cf. [4].

We consider now the space of polynomials and restrict ourselves for the moment to the one-dimensional case. A discrete distribution is defined as a Clifford-valued linear functional over this space, cf. [5]. We distinguish between two types of distributions. In analogy to regular distributions we associate with a discrete function $f$ a discrete distribution $F$ acting on a polynomial $V$ in the following way:

$$
\langle F, V \rangle := \sum_{x \in \mathbb{Z}} V(x) f(x)
$$

under the condition that the sum converges, $f$ is the so-called density function. In Remark 1 we explain more detailed the chosen distribution space.

Associating the discrete delta distribution $\delta_j$ with the discrete delta function $\delta_j$ applied to a homogeneous polynomial leads to

$$
\langle \delta_j, \xi^\ell[1] \rangle = \sum_{x \in \mathbb{Z}} \xi^\ell[1](x) \delta_j(x) = \xi^\ell[1](j), \ j \in \mathbb{Z}.
$$
Thus, distributions which possess a density function can be expressed by means of discrete delta distributions: \( F = \sum_{n \in \mathbb{Z}} \delta_n f(n) \). Since \( \delta_j[V] = V(j) \) we obtain

\[
\sum_{n \in \mathbb{Z}} \delta_n f(n) = \sum_{n \in \mathbb{Z}} V(n) f(n) = F[V].
\]

Singular distributions, i.e. distributions which cannot be assigned to a discrete function are represented using a sequence of its moments, see below.

For both types of distributions the action of the lowering and raising operators \( \Delta^\pm \) and \( X^\pm \) are given by

\[
\langle \Delta^\pm F, V \rangle = -\langle F, V \Delta^\mp \rangle \quad \text{and} \quad \langle X^\pm F, V \rangle = \langle F, VX^\mp \rangle,
\]

such that the skew Weyl relations are satisfied. Thus, the actions of the co-ordinate difference operator \( \partial \) and the co-ordinate vector variable \( \xi \) are also determined for discrete distributions:

\[
\langle \partial F, V \rangle = -\langle F, V (\Delta^- e^+ + \Delta^+ e^-) \rangle = -\langle F, V \partial^\dagger \rangle \quad \text{and} \quad \langle \xi F, V \rangle = \langle F, V (X^- e^- + X^+ e^+) \rangle = \langle F, V \xi^\dagger \rangle,
\]

where \( \partial^\dagger = e^- \Delta^+ + e^+ \Delta^- \) and \( \xi^\dagger = e^+ X^+ + e^- X^- \). Applying powers of difference operators \( \partial^k \) to \( \delta_j \)

\[
\langle \partial^k \delta_j, \xi^\ell [1] \rangle = (-1)^k \langle \delta_j, (\xi^\ell [1]) (\partial^\dagger)^k \rangle = \begin{cases} (-1)^k \frac{\ell!}{(\ell-k)!} \xi^{\ell-k} [1](j), & k \leq \ell \\ 0, & k > \ell \end{cases}
\]

leads to an equivalent representation of arbitrary discrete distributions by its dual Taylor series

\[
F = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial^k \delta_0 \langle F, \xi^k [1] \rangle
\]

where \( \{ \langle F, \xi^k [1] \rangle \} \) is the sequence of the moments of \( F \).

In higher dimensions the dual Taylor decomposition of \( F \) is given by, cf. [7]:

\[
F = \sum_{\alpha_n=0}^{\infty} \cdots \sum_{\alpha_1=0}^{\infty} \frac{(-1)^{\alpha_1+\cdots+\alpha_n}}{\alpha_1! \cdots \alpha_n!} \partial_n^{\alpha_n} \cdots \partial_1^{\alpha_1} \delta_0 \langle F, \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} [1] \rangle.
\]

**Remark 1.** In general distributions can be defined as linear functionals, that is as continuous, Clifford-valued linear forms on spaces of functions on \( \mathbb{Z} \). In particular the topological dual of compactly supported functions on \( \mathbb{Z} \) corresponds to the space of all functions on \( \mathbb{Z} \). More general one can consider spaces of functions with weight conditions of any type, e.g. rapidly decreasing sequences leading to tempered distributions, cf. [5]. However, we define distributions as linear functionals on the space of polynomials. The reason therefor is, that this space contains not only regular distributions. But each distribution admits a dual Taylor series expansion in terms of the derivatives of the discrete delta distribution \( \partial^k \delta \) what we will use to represent the Gaussian distribution.
3 The Gaussian distribution

As we will show later on, there is an intrinsic connection between the fundamental solution of the discrete Heat equation and the discrete Gaussian distribution. Therefore, we first discuss this discrete Gaussian distribution in detail in the present section. We begin to introduce the one-dimensional discrete Gaussian distribution and continue with the generalization to higher dimensions.

3.1 The one-dimensional discrete Gaussian distribution

Similar to the continuous setting where the Gaussian distribution acts on the classical homogeneous polynomials $x^\ell$ according to

$$\int_{x\in\mathbb{R}} x^\ell \exp\left(\frac{-x^2}{2}\right) \, dx = \begin{cases} 0, & \ell \text{ odd} \\ (\ell - 1)!! \sqrt{2\pi}, & \ell \text{ even} \end{cases},$$

we introduce a discrete Gaussian distribution $G$ by defining its action on the discrete homogeneous polynomials $\xi^\ell[1]$:

$$G[\xi^\ell[1]] := \begin{cases} \sqrt{2\pi}, & \ell = 0 \\ 0, & \ell \text{ odd} \\ (\ell - 1)!! \sqrt{2\pi}, & \ell \text{ even}. \end{cases}$$

Since the sequence of moments $\{G[\xi^\ell[1]]\}_\ell$ of the distribution $G$ is known, we can express $G$ in its dual Taylor series:

$$G = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial^\ell \delta_0 \, G[\xi^\ell[1]].$$

Using $(2k-1)!! = \frac{(2k-1)!}{2^{k-1}(k-1)!}$, the representation of $G$ by its dual Taylor series then is given by

$$G = \sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{2^k k!} \partial^{2k} \delta_0 = \sqrt{2\pi} \exp\left(\frac{\partial^2}{2}\right) \delta_0.$$

One can recognize the exponential which also appears in classical Gaussian distributions. However, now the exponent is a lowering operator $\partial$ instead of the usual raising operator.

**Remark 2.** Given the apparent similarity to the continuous setting, one could consider the discrete function $\exp\left(-\frac{\xi^2}{2}\right)[1]$ and examine whether it can be used as a density function. Note that the order of operations in this expression is important, since $\left(\xi^2[1]\right)^k$ does not equal $\xi^{2k}[1]$. This function is the formal analogue of the continuous density function of the continuous Gaussian distribution, as may be seen also from the fundamental property

$$\partial \exp\left(-\frac{\xi^2}{2}\right)[1] = -\xi \exp\left(-\frac{\xi^2}{2}\right)[1].$$
However, it is not scalar-valued because the discrete homogeneous powers $\xi^\ell[1]$ are not scalar-valued, and even more importantly, for this function to be a density function of a (regular) discrete distribution, it should have compact support or at least vanish at infinity. However, neither of these requirements is fulfilled, hence it is not suitable as density function of a discrete Gaussian distribution.

The distribution $G$ has a discrete density function since one can rewrite it as follows

$$G = \sum_{k=0}^{\infty} \sqrt{\frac{2\pi}{2k^k}} \sum_{i=-k}^{k} (-1)^{i+k} \left( \frac{2k}{i+k} \right) \delta_i = \sqrt{2\pi} \sum_{p \in \mathbb{Z}} \left[ \sum_{k=|p|}^{\infty} (-1)^{p+k} \frac{(2k)^{p+k}}{2k^k} \right] \delta_p$$

$$= \sum_{p \in \mathbb{Z}} \frac{\sqrt{2\pi}}{e} \text{BesselI}(p, 1) \delta_p$$

with Bessel the modified Bessel function of the first kind. This density function is depicted in Figure 1.

**Lemma 1.** The discrete Gaussian distribution satisfies in distributional sense: $\xi G = -\partial G$.

**Proof.** The co-ordinate difference $\partial$ acting on the Gaussian distribution $G$ gives

$$\partial G = \sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{(2k)!} (2k-1)!! \partial^{2k+1} \delta_0.$$ 

The skew Weyl relation $\xi \partial = \partial \xi - 1$ then implies that $\xi \partial^{2k} = \partial^{2k} \xi - 2k \partial^{2k-1}$, combined with $\xi \delta_0 = 0$ leads to

$$\xi G = \sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{(2k)!} (2k-1)!! (-2k \partial^{2k-1}) \delta_0 = -\sum_{k=1}^{\infty} \frac{\sqrt{2\pi}}{(2k-2)!} (2k-3)!! \partial^{2k-1} \delta_0$$

$$= -\sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{(2k)!} (2k-1)!! \partial^{2k+1} \delta_0 = -\partial G$$

proving the lemma. 

**Remark 3.** The previous property only holds in distributional sense. Since the action of $\xi$ on the delta function $\delta_0$ is not zero (unlike its action on the delta distribution $\delta_0$), this property does not hold for the density function associated with the discrete Gaussian distribution.

### 3.2 The higher dimensional discrete Gaussian distribution

Following the strategy developed in the one-dimensional case, we now define a discrete Gaussian distribution $G$ in general dimension $m$, as

$$G = \sum_{k_1, \ldots, k_m=0}^{\infty} \frac{(\sqrt{2\pi})^m}{2^{k_1+\ldots+k_m} k_1! \ldots k_m!} \partial_{2k_1} \ldots \partial_{2k_m} \delta_0 = (2\pi)^{\frac{m}{2}} \exp \left( \frac{\partial^2}{2} \right) \delta_0$$
with discrete density function
\[ g(x_1, \ldots, x_m) = \left(\frac{\sqrt{2\pi}}{e}\right)^m \text{BesselI}(x_1, 1) \ldots \text{BesselI}(x_m, 1). \]

The action of the co-ordinate vector variable \( \xi_j \) on this distribution follows the traditional rule \( \xi_j G = -\partial_j G \), from which it follows that
\[
\xi G = -\partial G \quad \text{and} \quad EG = \sum_{j=1}^{m} \xi_j \partial_j G = -\sum_{j=1}^{m} \xi_j^2 G = -\xi^2 G.
\]

Note that here the order of the co-ordinate difference operators \( \partial_{m_k}^2, \ldots, \partial_{1_k}^2 \) doesn’t matter since the even powers ensure that they are scalar-valued.

## 4 A fundamental solution of the discrete Heat equation

Consider a discrete version of the Heat equation, given by
\[
(\Delta^* - \partial_t) u(x, t) = 0, \quad x \in \mathbb{Z}^m, \quad t \in \mathbb{R}^+,
\]
with \( \Delta^* \) the discrete Laplacian, i.e. we consider the case where space is discrete and time continuous. The method to determining solutions of the Heat equation with a given initial temperature, is based on the notion of a fundamental solution \( G_t \) of the Heat equation, satisfying in distributional sense
\[
(\Delta^* - \partial_t) G_t(x, t) = \delta(t) \delta_0
\]
where \( \delta(t) \) is the continuous delta distribution in the time variable and \( \delta_0 \) the discrete delta distribution in the space variable.

Any discrete distribution can be written as a dual Taylor series. Based on the formal equivalence to the continuous space-time setting, we consider a distribution which only involves even powers of differences acting on the discrete delta distribution: \( \partial^{2k} \delta_0 \), i.e. we propose the following form for the fundamental solution:
\[
G_t = \sum_{k=0}^{\infty} c_k(t) \partial^{2k} \delta_0
\]
with the continuous functions \( c_k(t) \) yet to be determined, in such a way that
\[
(\partial_t - \Delta^*) G_t = \delta(t) \delta_0.
\]
Therefore, we determine both \( \partial_t G_t \) and \( \Delta^* G_t \):
\[
\Delta^* G_t = \sum_{k=0}^{\infty} c_k(t) \partial^{2k+2} \delta_0
\]
\[
\partial_t G_t = c'_0(t) \delta_0 + \sum_{k=1}^{\infty} c'_k(t) \partial^{2k} \delta_0 = c'_0(t) \delta_0 + \sum_{s=0}^{\infty} c'_{s+1}(t) \partial^{2s+2} \delta_0.
\]
In order for (2) to be fulfilled, it must hold that

\[ c'_0(t) \delta_0 + \sum_{k=0}^{\infty} (c'_{k+1}(t) - c_k(t)) \partial^{2k+2} \delta_0 = \delta(t) \delta_0 \]

or equivalently \( c'_0(t) = \delta(t) \) and \( c'_{k+1}(t) = c_k(t) \). We thus see that putting \( c_0(t) = H(t) \), \( c_1(t) = t H(t) \), \ldots, \( c_k(t) = \frac{t^k}{k!} H(t) \) with \( H(t) \) the continuous Heaviside, ensures that

\[ G_t = H(t) \sum_{k=0}^{\infty} \frac{t^k}{k!} \partial^{2k} \delta_0 = H(t) \exp \left( t \partial^2 \right) \delta_0 \]

is a fundamental solution of the discrete Heat equation. It consists of continuous distributions in \( t \) combined with discrete distributions in \( x \).

**Remark 4.** This definition of the discrete fundamental solution resembles (up to a constant) the discrete Gaussian distribution, and is in this way formally equivalent to the fundamental solution of the continuous Heat equation.

### 4.1 A density function of the fundamental solution \( G_t \)

The fundamental solution \( G_t \) can be rewritten in function of the discrete delta distributions \( \delta_n, n \in \mathbb{Z} \):\[
G_t = H(t) \sum_{k=0}^{\infty} \frac{t^k}{k!} \partial^{2k} \delta_0 = H(t) \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[ \sum_{i=-k}^{k} (-1)^{i+k} \left( \frac{2k}{i+k} \right) \delta_i \right]
\]

\[
= H(t) \sum_{n \in \mathbb{Z}} (-1)^n \left[ \sum_{k=|n|}^{\infty} \frac{(-1)^k t^k}{k!} \left( \frac{2k}{n+k} \right) \right] \delta_n.
\]

If

\[
\sum_{k=|n|}^{\infty} \frac{(-1)^k t^k}{k!} \left( \frac{2k}{n+k} \right)
\]

converges for all \( n \in \mathbb{Z} \), the discrete function

\[ g := (x,t) \rightarrow (-1)^n \sum_{k=|n|}^{\infty} \frac{(-1)^k t^k}{k!} \left( \frac{2k}{n+k} \right) H(t) \]

is said to be the density function of the distribution \( G_t \). The infinite sum converges for all \( n \in \mathbb{Z} \). Indeed,

\[
\sum_{k=|n|}^{\infty} \frac{(-1)^n+k t^k}{k!} \left( \frac{2k}{n+k} \right) = \text{BesselI}(|x|, 2t) e^{2t}
\]

with \( \text{BesselI} \) the modified Bessel function of the first kind. This density function is depicted in Figure 2 for several values of \( t \). Note that for \( t = 0 \), this density function is the discrete delta function \( \delta_0 \).
5 Discrete Convolution Theory

**Definition 1.** Consider two discrete functions $f$ and $g$ and define their convolution $f * g$ as the discrete function

$$(f * g)(n) = \sum_{x \in \mathbb{Z}} f(x)g(n-x) = \sum_{m \in \mathbb{Z}} f(n-m)g(m).$$

For this convolution to exist, it suffices that the intersection of the support of $f$ with the support of $g(n-\cdot)$ is compact, which is certainly the case if one of the two discrete functions has a compact support. For example, consider the case where $\text{supp}(f) = \text{supp}(g) = \mathbb{Z}^+$. The intersection of $\text{supp}(f)$ with $\text{supp}(g(n-\cdot))$ is always compact for every $n \in \mathbb{Z}$.

**Remark 5.** Note that this convolution is only symmetric if $f$ and $g$ commute, for example if either of the two functions $f$ or $g$ is scalar. This is not necessarily true for all discrete functions, as one can see from the example $f = e^+\delta_0$ and $g = e^-\delta_0$ which gives:

$$(f * g)(n) = \sum_{x \in \mathbb{Z}} f(x)g(n-x) = e^+e^-\delta_0(n)$$

$$(g * f)(n) = \sum_{x \in \mathbb{Z}} g(x)f(n-x) = e^-e^+\delta_0(n).$$

**Lemma 2.** Given two discrete functions $f$ and $g$ and a Clifford constant $a$, it holds that

$$(af) * g = a(f * g), \quad f * (ag) = (f a) * g, \quad f * (ga) = (f * g)a.$$

**Lemma 3.** Given two discrete functions $f$ and $g$, the convolution has the following property:

$$(f\partial) * g = f * (\partial g).$$

**Proof.**

$$(f\partial) * g(n) = \sum_{x \in \mathbb{Z}} (f\partial)(x)g(n-x)$$

$$= \sum_{x \in \mathbb{Z}} \left( f(x+1)e^+ - f(x)(e^+ - e^-) - f(x-1)e^- \right)g(n-x)$$

$$= \sum_{y \in \mathbb{Z}} f(y) \left( e^+g(n-(y-1)) - (e^+ - e^-)g(n-y) - e^-g(n-(y+1)) \right)$$

$$= \sum_{y \in \mathbb{Z}} f(y)(\partial g)(n-y) = f * (\partial g)(n).$$

$\square$
Example 1. The convolutions of discrete functions $\partial^k \delta$ with compact support with a discrete function $f$ are given by

$$(\partial^k \delta \ast f) (n) = (\partial^k f) (n).$$

Indeed, $(\delta \ast f) (n) = \sum_{x \in \mathbb{Z}} \delta(x) f(n-x) = f(n)$ and

$$(\partial^2 g) \ast h (n) = (g \partial^2) \ast h (n) = (g \ast (\partial^2 h)) (n)$$

and for $g$ scalar (in this case: $g = \partial^{2\ell} \delta$ for some $\ell$),

$$(\partial g \ast h) (n) = (g \partial h) (n) = (g \ast (\partial h)) (n).$$

On the other hand, by using lemma 3, one can determine that

$$(f \ast \partial^k \delta) (n) = ((f \partial^k) \ast \delta) (n) = (f \partial^k) (n).$$

Remark 6. For a discrete function $f$, the convolution with $\partial^k \delta_0$ is not necessarily symmetric. For example, choose $f = e^+ \delta_0$ then

$$\partial \delta_0 \ast f = (\partial \delta_0 \ast \delta_0) e^+ = \partial \delta_0 e^+ = e^- (\delta_0 - \delta_1)$$

$$f \ast \partial \delta_0 = (\delta_0 e^+) \ast \partial \delta_0 = e^+ \partial \delta_0 = e^+ e^- (\delta_0 - \delta_1).$$

However, the convolution with $\partial^k \delta_0$ is symmetric if $k$ is even or $f$ scalar valued.

Definition 2. Consider two regular distributions $T_f$ and $T_g$ with discrete density functions $f$, respectively $g$, at least one of them having compact support. The convolution of $T_f$ and $T_g$ is the distribution denoted $T_f \ast T_g$ which acts as follows on discrete polynomials:

$$\langle T_f \ast T_g, V \rangle = \left<T_g(y), \left<T_f(x), V(x+y) \right> \right> = \sum_{y \in \mathbb{Z}} \left( \sum_{x \in \mathbb{Z}} V(x+y) f(x) \right) g(y).$$

Remark 7. Let $f$ and $g$ be discrete density functions of the distributions $T_f$ and $T_g$, at least one of them having compact support. Since

$$\langle T_f \ast T_g, V \rangle = \sum_{y \in \mathbb{Z}} \left( \sum_{x \in \mathbb{Z}} V(x+y) f(x) \right) g(y) = \sum_{x \in \mathbb{Z}} V(x) \left( \sum_{y \in \mathbb{Z}} f(y) g(x-y) \right)$$

$$= \sum_{x \in \mathbb{Z}} V(x) (f \ast g) (x) = \langle T_{f \ast g}, V \rangle,$$

we see that $T_f \ast T_g = T_{f \ast g}$.

Example 2. The convolution of the discrete distributions $\partial^k \delta_0$ with a regular distribution $T_f$ with density function $f$ satisfies

$$\partial^k \delta \ast T_f = \partial^k T_f \quad \text{and} \quad T_f \ast \partial^k \delta = T_f \partial^k.$$
Indeed
\[
\langle \partial^k \delta \ast T_f, V \rangle = \left\langle T_f(x), \left(\partial^k \delta(y), V(x + y)\right) \right\rangle = \sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} V(x + y) \partial_y^k \delta(y)\right) f(x)
\]
\[
= (-1)^k \sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} \left(V(\partial_y^k)\right)(x + y) \delta(y)\right) f(x)
\]
\[
= (-1)^k \sum_{x \in \mathbb{Z}} \left(V(\partial_y^k)\right)(x) f(x) = (-1)^k \langle T_f, V(\partial^k)\rangle
\]
\[
= \langle \partial^k T_f, V \rangle.
\]
On the other hand, consider \( T_f \ast \partial^k \delta \):
\[
\langle T_f \ast \partial^k \delta, V \rangle = \left\langle \partial^k \delta(x), \left(T_f(y), V(x + y)\right) \right\rangle = \sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} V(x + y) f(y)\right) \partial_y^k \delta(x)
\]
\[
= (-1)^k \sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} V(x + y) f(y)\right) (\partial_y^k) \delta(x).
\]
Note that
\[
\left(\sum_{y \in \mathbb{Z}} V(x + y) f(y)\right) \partial_y^1 \\
= \sum_{y \in \mathbb{Z}} \left(V(x + y + 1) f(y) e^- - V(x + y) f(y) (e^- e^+) - V(x + y - 1) f(y) e^+\right)
\]
\[
= \sum_{y \in \mathbb{Z}} V(x + y) \left(f(y - 1) e^- - f(y) (e^- e^+) - f(y + 1) e^+\right)
\]
\[
= - \sum_{y \in \mathbb{Z}} V(x + y) (f \partial_y)(y)
\]
and hence
\[
\langle T_f \ast \partial^k \delta, V \rangle = \sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} V(x + y) (f \partial_y^k)(y)\right) \delta(x) = \sum_{y \in \mathbb{Z}} V(y) (f \partial^k_y)(y) = \langle T_f \partial^k, V \rangle.
\]
To define the convolution of the distribution \( \partial^k \delta_0 \) and a general distribution \( G \), we write down \( G \) as its dual Taylor series
\[
G = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial^k \delta_0 c_k, \text{ with } c_k = G \left[ \xi^k[1] \right].
\]
The distribution \( \partial^k \delta_0 \ast G \) is then given by
\[
\partial^k \delta_0 \ast G = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\partial^k \delta_0 \ast \partial^k \delta_0 \right) c_k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial^{k+k} \delta_0 c_k = \partial^k G.
\]
If \( G \) is regular with density function \( g \), i.e. \( G = T_g \), the result indeed corresponds with the first definition.

**Lemma 4.** The derivative \( \partial \) of a convolution of two regular distributions \( T_f \) and \( T_g \) is the convolution of the distribution \( \partial T_f \) and \( T_g \):

\[
\partial (T_f \ast T_g) = (\partial T_f) \ast T_g.
\]

**Proof.** When \( T_f \) is a derivative of the delta distribution, i.e. \( T_f = \partial^k \delta_0 \), one easily sees that

\[
\partial (\partial^k \delta_0 \ast T_g) = \partial (\partial^k T_g) = \partial^{k+1} T_g = \partial^{k+1} \delta_0 \ast T_g.
\]

For a general distribution \( T_f \) we see that

\[
\langle \partial (T_f \ast T_g), V \rangle = - \langle T_f \ast T_g, V \partial \rangle = - \langle T_g, \langle T_f, V \partial \rangle \rangle = \langle T_g, \langle \partial T_f, V \rangle \rangle = \langle (\partial T_f) \ast T_g, V \rangle.
\]

**Example 3.** Consider the convolution \( G \) of the (discrete part of the) fundamental solution \( G_t \) with a regular distribution \( T_f \) with density function \( f \):

\[ G := \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \partial^{2k} \delta_0 \right) \ast T_f = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \partial^{2k} T_f \right). \]

This is a regular distribution with density function \( g(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \partial^{2k} f(x) \).

### 6 Solutions of the discrete Heat equation

Consider the problem \( (\partial_t - \Delta^*) u(x,t) = f(x,t) \) where \( f(x,t) \) is a given function. The solution \( u(x,t) \) can be found in the following way:

1. Determine the convolution \( FS \ast f =: g(x,t) \) of the (density function of the) fundamental solution \( FS \) and the function \( f \). If \( f \) is a density function of a regular distribution \( T_f \), the function \( g(x) \) is then the density function of the distribution \( FS \ast T_f =: G \), which will satisfy in distributional sense \( (\partial_t - \Delta^*) G = T_f \). Indeed, lemma 4 combined with the classical property of the derivative of a continuous convolution shows that

\[
(\partial_t - \Delta^*) G = (\partial_t - \Delta^*) (FS \ast T_f) = ((\partial_t - \Delta^*) FS) \ast T_f = \delta \ast T_f = T_f.
\]

hence \( g(x,t) \), which is the density function of \( G \) is the suitable solution of the considered problem.

2. Note that the convolution \( FS \ast f \) is a combination of a discrete and a continuous convolution, i.e. if \( f \) is a function in the discrete variable \( x \) and the continuous
time variable \(s\), then \(FS * f\) is a distribution in the discrete variable \(y\) and the continuous variable \(s\) given by

\[
(FS * f) (y, s) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} H(t) t^k (\partial^{2k} \delta \ast f) (y, s - t) \, dt
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} H(t) t^k (\partial^{2k} f) (y, s - t) \, dt
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \left( H(t) t^k \ast f \right) (y, s)
\]

where we first considered the discrete convolution and second the continuous convolution with respect to the time-variable. Although the discrete convolution \(\partial^{2k} \delta \ast f\) will always exist, one still has to check the convergence of the series in \(t\):

\[
\sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} H(t) t^k (\partial^{2k} f) (y, s - t) \, dt.
\]

**Example 4.** Consider the function \(f(x, t) = \chi(0, +\infty)(t) \delta(x)\), which is the example of ‘solution in one point’. The convolution

\[
FS \ast f = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \int_{-\infty}^{\infty} H(t) t^k \chi(0, +\infty)(s - t) \, dt \right) (\partial^{2k} \delta \ast \delta)
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \chi(0, +\infty)(s) \int_{0}^{\infty} t^k \, dt \right) \partial^{2k} \delta = \chi(0, +\infty)(s) \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{t^{k+1}}{(k+1)!} \right]_0^s \partial^{2k} \delta
\]

\[
= \chi(0, +\infty)(s) \sum_{k=0}^{\infty} \frac{s^{k+1}}{(k+1)!} \partial^{2k} \delta.
\]

We now consider the convergence of this function \(g(x, s)\):

\[
\chi(0, +\infty)(s) \sum_{k=0}^{\infty} \frac{s^{k+1}}{(k+1)!} \partial^{2k} \delta = \chi(0, +\infty)(s) \sum_{k=0}^{\infty} \frac{s^{k+1}}{(k+1)!} \sum_{j=-k}^{k} (-1)^{k+j} \binom{2k}{k+j} \delta_j
\]

\[
= \chi(0, +\infty)(s) \sum_{p \in \mathbb{Z}} \left( \sum_{k=|p|}^{\infty} \frac{(-1)^{k+p} s^{k+1}}{(k+1)!} \binom{2k}{k+p} \right) \delta_p
\]

whence we see that

\[
g(x, s) = \chi(0, +\infty)(s) \sum_{k=|x|}^{\infty} \frac{(-1)^{k+x} s^{k+1}}{(k+1)!} \binom{2k}{k+x}.
\]

One can check, for example with Maple, that

\[
g(x, s) = \frac{s^{1+|x|}}{\Gamma(1+|x|)} \text{hypergeom}\left(\left[1 + |x|, |x| + \frac{1}{2}\right], \left[\frac{3}{2} |x| + \frac{3}{2} \frac{1}{|x| - 1}, \frac{3}{2} |x| + \frac{3}{2} + \frac{1}{2} |x| - 1\right], -4s\right).
\]
We compare this solution to the continuous setting and we consider the problem
\[
\begin{cases}
  u(x,0) = 0, & x \in \mathbb{R} \\
  (\partial_t - \partial_x^2) u(x,t) = \delta(x), & x \in \mathbb{R}, \ t > 0.
\end{cases}
\]

There, the solution \( u(x,t) \) is given by the convolution of \( f(x,t) = \delta(x) \) with the continuous fundamental solution \( \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \), in this case resulting in
\[
u(y,s) = \int_0^s \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} f(y-x,s-t) \, dx \, dt
= \int_0^s \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \delta(y-x) \, dx \, dt
= \int_0^s \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} \, dt.
\]

Both the discrete and continuous solution are depicted in Figure 3.

**Example 5.** When considering the Heat problem \( (\partial_t - \Delta^*) u(x,t) = f(x,t) \) with \( f(x,t) = \delta(x) \delta(t) \), for example a so-called 'smoke bomb', we end up with our fundamental solution \( FS \). Indeed, the convolution of the fundamental solution with \( \delta(x) \delta(t) \) results again in our fundamental solution:
\[
FS * f = \left( H(t) \sum_{k=0}^{\infty} \frac{t^k}{k!} \partial^{2k} \delta \right) * \delta(x) \delta(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \int_{-\infty}^{\infty} H(t) s^k \delta(s-t) \, dt \right) \partial^{2k} \delta(t)
= H(s) \sum_{k=0}^{\infty} \frac{1}{k!} s^k \partial^{2k} \delta = FS.
\]

### 7 Heat polynomials

The continuous Heat polynomials \( p_{\beta}(x,t) \) are polynomial solutions to the Heat equation
\[
(\partial_t - \Delta) u(x,t) = 0, \quad x \in \mathbb{R}, \ t > 0
\]
with initial condition \( p_{\beta}(x,0) = x^\beta \). For integer values of \( \beta \), we get the following solutions:
\[
p_n(x,t) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2k} t^k}{(n-2k)! k!}.
\]
The Heat polynomials are suitable to determine in general solutions of the Heat equation with a given initial condition.
7.1 Construction

In our setting of one discrete space variable \( x \) and a continuous time variable \( t \), we search for analogous discrete polynomial solutions \( p_n(x,t) \) of the discrete Heat equation

\[
(\partial_t - \Delta^*) u(x,t) = 0, \quad x \in \mathbb{Z}, \ t > 0
\]

with initial condition \( p_n(x,0) = \xi^n[1] \).

To this end, we first let the Fourier transform \( \mathcal{F}_x \) in the variable \( x \), act on the Heat equation. We can decompose the function \( u(x,t) \) in its Taylor series

\[
u(x,t) = \sum_{k=0}^{\infty} \xi^k[1] c_k(t)\]

with \( c_k(t) \) functions of the continuous variable \( t \). The Fourier transform of \( u(x,t) \) is then the discrete distribution \( \mathcal{F}_x[u] = \hat{u} \) given by the dual Taylor series

\[
\hat{u} = \sum_{k=0}^{\infty} \partial^k \delta_0 c_k(t).
\]

Since for any discrete function \( f \), the Fourier transform of the derivative of \( f \) is minus the operator \( \xi \) acting on the Fourier transform of \( f \):

\[
\mathcal{F}_x[\partial f] = -\xi \mathcal{F}_x[f],
\]

we see that \( \mathcal{F}_x[u] = \hat{u} \) must satisfy (in distributional sense):

\[
(\partial_t - \xi^2) \hat{u} = 0
\]

or thus also

\[
\partial_t \left( e^{-\xi^2 t} \hat{u} \right) = 0.
\]

The boundary condition then implies that \( e^{-\xi^2 t} \hat{u} = \mathcal{F}[\xi^n[1]] \) or thus \( \hat{u} = e^{\xi^2 t} \mathcal{F}[\xi^n[1]] \).

Since \( \mathcal{F}[\xi^k[1]] = \partial^k \delta_0 \), we can hence find the appropriate solution \( u(x,t) \).

**Example 6.** As a first trivial example, consider the problem

\[
\begin{cases}
(\partial_t - \Delta^*) u(x,t) = 0, \quad x \in \mathbb{Z}, \ t > 0 \\
u(x,0) = \phi(x), \quad \phi(x) = 1.
\end{cases}
\]

Then \( \hat{\phi} = \delta_0 \) and \( \hat{u} = e^{\xi^2 t} \delta_0 = \delta_0 \) since \( \xi \delta_0 = 0 \). We’re thus looking for a discrete function \( u(x,t) \) such that \( \hat{u} = \delta_0 \). We may conclude that \( u(x,t) = 1 \).

**Example 7.** A more illustrative example is given by considering the problem

\[
\begin{cases}
(\partial_t - \Delta^*) u(x,t) = 0, \quad x \in \mathbb{Z}, \ t > 0 \\
u(x,0) = \phi(x), \quad \phi(x) = \xi^k[1].
\end{cases}
\]

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Then \( \hat{\phi} = \partial^{2k} \delta_0 \) and

\[
\hat{u} = e^{\xi^2 t} \partial^{2k} \delta_0 = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \xi^{2\ell} (2k)! \partial^{2k-2\ell} \delta_0
\]

since \( \xi \partial^k \delta_0 = -k \partial^{k-1} \delta_0 \). We are thus looking for a discrete function \( u(x,t) \) such that

\[
\hat{u} = \sum_{\ell=0}^{k} \frac{t^\ell}{\ell!} (2k)! (2k-2\ell)! \partial^{2k-2\ell} \delta_0.
\]

We may conclude that

\[
u(x,t) = \sum_{\ell=0}^{k} \frac{t^\ell}{\ell!} (2k)! (2k-2\ell)! \xi^{2k-2\ell+1}[1],
\]

which formally resembles the continuous solutions.

Example 8. On the other hand, for the discrete Heat polynomials of odd degree, i.e. discrete polynomial solutions to the problem

\[
\begin{cases}
(\partial_t - \Delta^*) u(x,t) = 0, & x \in \mathbb{Z}, \ t > 0 \\
u(x,0) = \phi(x), & \phi(x) = \xi^{2k+1}[1],
\end{cases}
\]

our discrete solution \( u(x,t) \) is given by

\[
u(x,t) = \sum_{\ell=0}^{k} \frac{t^\ell}{\ell!} (2k+1)! (2k-2\ell+1)! \xi^{2k-2\ell+1}[1].
\]

We conclude that the discrete Heat polynomials \( p_n(x,t) \) are given by

\[
p_{2k}(x,t) = \sum_{\ell=0}^{k} \frac{t^\ell}{\ell!} (2k)! (2k-2\ell)! \xi^{2k-2\ell}[1],
\]

\[
p_{2k+1}(x,t) = \sum_{\ell=0}^{k} \frac{t^\ell}{\ell!} (2k+1)! (2k-2\ell+1)! \xi^{2k-2\ell+1}[1]
\]

or equivalently

\[
p_n(x,t) = \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{t^\ell}{\ell!} \frac{n!}{(n-2\ell)!} \xi^{n-2\ell}[1].
\]

The first few discrete Heat polynomials are then given by

\[
P_0(x,t) = 1,
\]

\[
P_1(x,t) = \xi[1],
\]

\[
P_2(x,t) = \xi^2[1] + 2t,
\]

\[
P_3(x,t) = \xi^3[1] + 6t \xi[1],
\]

\[
P_4(x,t) = \xi^4[1] + 12t \xi^2[1] + 12t^2,
\]

\[
P_5(x,t) = \xi^5[1] + 20t \xi^3[1] + 60t^2 \xi[1].
\]

They are depicted in Figure 4.
Remark 8. When we determine the solution to the problem

\[ (\partial_t - \Delta^*) u(x, t) = \xi^{2k}[1] \delta(t) \]

with the usual method of convoluting with the fundamental solution, we get

\[
FS \ast (\xi^{2\ell}[1] \delta(t)) = \sum_{k=0}^{\infty} \left( \left( H(t) \frac{t^k}{k!} \right) \ast \delta(t) \right) \partial_{xx}^{2\ell} \xi^{2\ell}[1]
\]

\[
= \sum_{k=0}^{\ell} \left( \int_{-\infty}^{\infty} H(s) \frac{s^k}{k!} \delta(t-s)ds \right) \frac{(2\ell)!}{(2\ell - 2k)!} \xi^{2\ell-2k}[1]
\]

\[
= \sum_{k=0}^{\ell} \left( \int_{0}^{\infty} \frac{s^k}{k!} \delta(t-s)ds \right) \frac{(2\ell)!}{(2\ell - 2k)!} \xi^{2\ell-2k}[1]
\]

\[
= \chi[0, +\infty](t) \sum_{k=0}^{\ell} \frac{t^k}{k!} \frac{(2\ell)!}{(2\ell - 2k)!} \xi^{2\ell-2k}[1]
\]

In this way, we also see the appearance of the Heat polynomials.

7.2 Solutions of the Heat equation with given initial condition

By means of the discrete Heat polynomials, we can determine solutions \( u(x, t) \) of the following Heat problem:

\[
\begin{cases}
(\partial_t - \Delta^*) u(x, t) = 0, & x \in \mathbb{Z}, \ t > 0 \\
u(x, 0) = \phi(x).
\end{cases}
\]

Indeed, every discrete function \( \phi(x) \) can be developed into a discrete Taylor series. We give the example where the initial function \( \phi(x) \) is the discrete delta function \( \delta_0 \).

The discrete delta function \( \delta_0 \) can be expressed in terms of the discrete homogeneous polynomials \( \xi^k[1] \) as follows:

\[
\delta_0 = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell)!^2} \xi^{2\ell}[1] + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{\ell! (\ell+1)!} \xi^{2\ell+1}[1] (e^+ - e^-).
\]

Substituting every discrete polynomial \( \xi^k[1] \) by its corresponding Heat polynomial \( p_n(x, t) \), gives us a solution \( u(x, t) \), satisfying the Heat equation, with initial condition...
\[ u(x, 0) = \delta(x): \]

\[
\begin{align*}
  u(x, t) &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell)!^2} \left[ \sum_{s=0}^{\ell} \frac{t^s}{s!} (2\ell)! (2\ell - 2s)! \xi^{2\ell - 2s}[1] \right] \\
  &\quad + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{\ell!(\ell+1)!} \left[ \sum_{s=0}^{\ell} \frac{t^s}{s!} (2\ell + 1)! (2\ell - 2s + 1)! \xi^{2\ell - 2s + 1}[1] \right] (e^+ - e^-) \\
  &= \sum_{s=0}^{\infty} \left[ \sum_{\ell=s}^{\infty} \frac{(-1)^\ell}{(2s)!} \frac{t^{\ell-s}}{[\ell-s]!} \left( \frac{2\ell}{\ell} \right) \xi^{2s}[1] \right] \\
  &\quad + \sum_{s=0}^{\infty} \left[ \sum_{\ell=s}^{\infty} \frac{(-1)^{s+1}}{(2s+1)!} \frac{t^{\ell-s}}{[\ell-s]!} \left( \frac{2\ell + 1}{\ell} \right) \xi^{2s+1}[1] (e^+ - e^-) \right].
\end{align*}
\]

One can easily check that for \( t = 0 \):

\[
\begin{align*}
  u(x, 0) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \xi^{2s}[1] + \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!(s+1)!} \xi^{2s+1}[1] (e^+ - e^-) = \delta(x)
\end{align*}
\]

and that \((\partial_t - \Delta^*) u(x, t) = 0\). The function \(u(x, t)\) is depicted in Figure 5 for \(-5 \leq x \leq 5\) and \(t \leq 10\).

One can show that \(u(x, t)\) is the function \(e^{-2t} \text{BesselI}(x, 2t)\), which is nothing else than the density function of the fundamental solution of the Heat equation. We may thus conclude that the fundamental solution \(\frac{\text{BesselI}(x, 2t)}{e^{2t}}\) is a solution to the problem

\[
\begin{align*}
  (\partial_t - \Delta^*) u(x, t) &= 0, \quad x \in \mathbb{Z}, t > 0 \\
  u(x, 0) &= \delta(x).
\end{align*}
\]

The following example will show that convergence in \(t\) of the solution \(u(x, t)\) is not always obvious.

**Example 9.** We consider the problem

\[
\begin{align*}
  (\partial_t - \Delta^*) u(x, t) &= 0, \quad x \in \mathbb{Z}, t > 0 \\
  u(x, 0) &= e^{\xi^2[1]}.
\end{align*}
\]

The initial function \(\phi(x) = e^{\xi^2[1]} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \xi^{2\ell}[1]\) converges for each \(x \in \mathbb{Z}\). Following the usual method, we replace each \(\xi^{2\ell}[1]\) by the corresponding Heat polynomial

\[
  p_{2\ell}(x, t) = \sum_{s=0}^{\ell} \frac{t^s}{s! (2\ell - 2s)!} \xi^{2\ell - 2s}[1],
\]

\[20\]
and get the following solution \( u(x,t) \) of the given problem:

\[
u(x,t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} p_{2\ell}(x,t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left[ \sum_{s=0}^{\ell} \frac{t^s}{s! (2\ell-2s)!} \xi^{2\ell-2s}[1] \right] = \sum_{\ell=0}^{\infty} \left[ \sum_{p=0}^{\infty} \frac{1}{\ell!} \frac{t^{\ell-p}}{(\ell-p)! (2p)!} \right] \xi^{2p}[1]
\]

The series \( \sum_{\ell=p}^{\infty} \frac{1}{\ell!} \frac{t^{\ell-p}}{(\ell-p)! (2p)!} = \sum_{s=0}^{\infty} \frac{t^s (2s + 2p)!}{s! (s+p)! (2p)!} \) converges only for \( t < \frac{1}{4} \) for which it equals

\[
\frac{1}{p! (1-4t)^{p+\frac{3}{2}}}
\]

We thus see that

\[
u(x,t) = \sum_{p=0}^{\infty} \frac{1}{p!} \frac{1}{(1-4t)^{p+\frac{3}{2}}} \xi^{2p}[1]
\]

This series converges for all \( x \in \mathbb{Z} \), but only for \( t < \frac{1}{4} \).

8 Conclusion and further research

In this contribution, we have successfully applied the discrete function theory to the Heat equation, yielding a discretized Heat equation where space is discrete and time continuous. It turned out that a fundamental solution is given by the discrete Gaussian distribution:

\[
G_t = H(t) \sum_{k=0}^{\infty} \frac{t^k}{k!} \partial^2 \delta_0.
\]

We investigated some properties and exposed differences to the continuous Gaussian distribution. After introducing the discrete convolution we applied this theory on the inhomogeneous Heat equation. Finally we constructed solutions of the Cauchy problem for the Heat equation by means of Heat polynomials illustrated by several examples.

A topic for future research encompasses a further discretization of the Heat equation by considering the time variable to be discrete and studying \( (\Delta - \Delta^+ f(x,t) = 0 \) where \( \Delta^+ \) denotes the forward difference with respect to the time variable. Both fundamental solution and heat polynomials will be considered for this totally discrete heat equation and their connection with the corresponding heat polynomials in this paper (discrete space - continuous time).

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References


